Published by Institute of Physics Publishing for SISSA

Received: *August 10, 2005*
Accepted: *August 25, 2005*
Published: *September 6, 2005*

Magnetic charges in local field theory

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Abstract: Novel lagrangians are discussed in which (non-abelian) electric and magnetic gauge fields appear on a par. To ensure that these lagrangians describe the correct number of degrees of freedom, tensor gauge fields are included with corresponding gauge symmetries. Non-abelian gauge symmetries that involve both the electric and the magnetic gauge fields can then be realized at the level of a single gauge invariant Lagrangian, without the need of performing duality transformations prior to introducing the gauge couplings. The approach adopted, which was initially developed for gaugings of maximal supergravity, is particularly suited for the study of flux compactifications.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry, Supersymmetry and Duality, Supergravity Models.

Contents

1. Introduction

It is generally believed that electric and magnetic gauge fields cannot be described in terms of a single local lagrangian. While electric and magnetic fields are defined by field strengths which can be related in a local fashion, the underlying vector potentials, in general, do not satisfy such a relation (this paper deals with four-dimensional gauge theories, so that both electric and magnetic potentials are vector fields). In certain cases, for instance, when describing magnetic monopoles in terms of the electric vector potential, the latter cannot be single-valued. In the absence of charges, the Bianchi identities (which imply that the field strengths can be expressed in terms of vector potentials) and the field equations for these vector potentials take a similar form. Assuming that we are dealing with n vector potentials, we have n Bianchi identities and n field equations. Upon rotating these $2n$ equations (by a symplectic $2n$-by-$2n$ matrix) there is the option of selecting n independent linear combinations of them to be interpreted as Bianchi identities whose solutions lead to a different set of vector potentials. In terms of these vector potentials there exists a different lagrangian which gives rise to field equations and Bianchi identities for the field strengths that are linearly equivalent to the original ones. However, the two dual sets of gauge fields are not locally related and therefore they cannot appear simultaneously in a given local lagrangian. By the same token one cannot have a local coupling of the gauge fields to magnetic charges as the latter couple locally to the dual magnetic gauge fields.

Hence different lagrangians can describe the same set of field equations and Bianchi identities. The transformations governing these inequivalent lagrangians are known as electric/magnetic duality. Symmetries of the combined equations of motion and Bianchi identities are not necessarily realized at the level of the lagrangian and may involve a subgroup

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of the electric/magnetic duality transformations. This poses a problem when switching on (possibly nonabelian) gauge interactions, as the gauging must proceed through electric gauge fields. A coupling to the magnetic charges seems therefore only possible after applying an appropriate electric/magnetic duality transformation by which all relevant charges are converted to electric ones. In this paper we demonstrate in a rather general framework how nevertheless one can have couplings to the magnetic charges without first converting them to electric ones. Here we should stress that we restrict ourselves to electric/magnetic charges that are mutually local, so that problems of a more fundamental nature are avoided.

The situation described above has an analogue in space-time dimensions other than four, where electric/magnetic duality takes the form of a duality between vector and tensor gauge fields. In d space-time dimensions, antisymmetric gauge fields of rank-p are dual to antisymmetric gauge fields of rank $d-2-p$. So the dual gauge field of an electric vector potential is an antisymmetric gauge field of rank $d-3$. While the former couples naturally to an electrically charged particle, the latter couples to a magnetically charged brane of $(d-4)$-dimensional spatial extension. When attempting to switch on gauge interactions one encounters the same problem as noted above in the four-dimensional context. Namely, one has to convert all the gauge fields that are supposed to couple to the charges to (electric) vector potentials. Furthermore, because the remaining vector gauge fields must be neutral, one must convert the charged vector fields that do not belong to the adjoint representation of the gauge group, to tensor fields. While this seems to pose no problem of principle, the field content of the theory thus depends sensitively on the gauging, so that introducing a gauge group is no longer a matter of simply switching on a corresponding gauge coupling constant. This fact precludes any uniform treatment of the gaugings of these theories and furthermore thoroughly obscures the symmetry structure of the underlying ungauged theory.

Recently, in our study of gaugings of maximal supergravities in five space-time dimensions [1] we exploited a framework in which these conversions between vector and tensor fields are no longer necessary. Gaugings are encoded in a so-called embedding tensor which defines the embedding of the gauge group into the symmetry group of the ungauged theory. The symmetry structure of the latter remains completely manifest (although the full symmetry is broken by the embedding tensor) because both vector and tensor fields are present and assigned to representations of the symmetry group of the ungauged theory. The presence of an intricate set of vector-tensor gauge transformations ensures that the number of physical degrees of freedom remains the same. In [1] it was already noted how this approach can be applied to gaugings of maximal supergravity in various possible space-time dimensions. As was explained in [2], one is generically dealing with hierarchies of vector-tensor gauge fields and there exists an intriguing interplay between the group-theoretical assignment of the various tensor fields and their duality relation. Although this was primarily explained in the context of the $\mathrm{E}_{k(k)}$ duality groups of maximal supergravity, the mechanism is by no means restricted to supergravity and can be applied to generic gauge theories. This paper explains how this is done in the context of four space-time dimensions, where the precise implementation of the mechanism was yet unknown.

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As discussed in [2], the gauge theory in four space-time dimensions is augmented by rank-2 tensor fields transforming in the adjoint representation of the symmetry group of the ungauged theory (in this case $E_{7(7)}$, as this is the symmetry group of the ungauged maximal supergravity). The analysis was based on the group-theoretical properties of the embedding tensor, which we will have to determine in the more general setting, as shall be discussed in due course. The presence of these tensor fields is reminiscent of a similar situation in the context of $N = 2$ supergravity. In [3] it was noted that flux compactifications of type-II theories on Calabi-Yau threefolds gave rise to a gauged $N = 2$ supergravity with electric and magnetic charges, which was not of the 'canonical type'. The presence of the tensor field is by itself not surprising in the context of a compactification from higher-dimensional supergravity. Because the gauging was abelian its effect was confined to the interactions of the tensor field with the vector gauge fields (apart from a scalar potential required by supersymmetry). In that particular case the fluxes are such that the theory remains symplectically invariant. The term symplectic 'invariance' is perhaps somewhat misleading. Rephrasing the latter result in the context of our work [4, 1], it means that the embedding tensor is parametrized in terms of $2n$ charges, which, when treated as spurionic quantities, preserve the manifest symplectic invariance. This 'spurionic' approach was a crucial ingredient of the group-theoretical analysis of [4, 1]. The symplectic invariance is thus an equivalence relation between two theories, rather than an invariance.

As we pointed out previously, we will be dealing with charges that are mutually local so that they can be converted to electric ones. In principle there is nothing wrong in having to perform a series of field dualities prior to switching on the charges. However, doing so will always obscure the symmetry structure that the theory has inherited from the ungauged theory. Moreover, the lagrangian often contains terms that diverge in the limit of vanishing gauge coupling constant, and there are also a number of practical drawbacks. When performing a symplectic reparametrization on the charges in the compactification discussed in [3], the theory can be brought in the more 'canonical' form that is known for $N = 2$ supergravity [5–9]. However, the manifest symplectic invariance is lost in that case. In the approach that we discuss in this paper, all of this is no longer necessary and moreover one can also discuss non-abelian gauge groups. In the formulation that we will present there is a topological term that takes a universal form encoded in terms of the embedding tensor and depending only on the vector and tensor fields. The lagrangian is fully gauge invariant, irrespective of whether the original rigid invariance was respected by the initial lagrangian, or only by the combined field equations and Bianchi identities. When imposing a gauge choice and integrating out certain fields (which originate from other parts in the lagrangian) the universal features are lost and a large variety of lagrangians is generated.

This paper is organized as follows. In section 2 we briefly discuss the issue of electric/magnetic duality. The gaugings with both electric and magnetic vector potentials is introduced in section 3, where we present the constraints on the embedding tensor and explain the connection with the tensor fields transforming in the adjoint representation of the symmetry group of the ungauged theory. In section 4 we derive the universal gauge invariant interactions for the vector-tensor system, which are fully encoded in terms of the embedding tensor. In section 5 we elucidate some of our results. We demonstrate how all tensor fields can be integrated out from the lagrangian in the presence of non-abelian gauge interactions, which amounts to effecting an electric/magnetic duality transformation in the presence of charges at the lagrangian level. We also discuss some features relevant to abelian gaugings and to the application of our formalism to gauged $N = 2$ supergravity. Undoubtedly our result will have many other applications with or without supersymmetry, but we refrain from presenting further explicit examples here.

2. Electric/magnetic duality

In the absence of charges, gauge invariant lagrangians in four space-time dimensions based on abelian gauge fields $A_\mu{}^\Lambda$, labeled by the index $\Lambda = 1, \ldots, n$, can be expressed in terms of their abelian field strengths, $\mathcal{F}_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu} A_{\nu]}{}^\Lambda$. The field equations for these fields and the Bianchi identities for the field strengths constitute $2n$ equations,

$$\partial_{[\mu}\mathcal{F}_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu}\mathcal{G}_{\nu\rho]\,\Lambda}\,, \tag{2.1}$$

where

$$\mathcal{G}_{\mu\nu\,\Lambda} = -\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial\mathcal{F}_{\rho\sigma}{}^\Lambda}\,. \tag{2.2}$$

Here we use a metric with signature $(-,+,+,+)$ and $\varepsilon_{0123} = 1$. The discussion below is valid for any space-time metric, and to simplify matters we will henceforth suppress $g_{\mu\nu}$ and restrict ourselves to flat Minkowski space. Obviously the set of equations (2.1) are invariant under rotations of the $2n$-component array $(\mathcal{F}^\Lambda, \mathcal{G}_\Lambda)$,

$$\begin{pmatrix} \mathcal{F}^\Lambda \\ \mathcal{G}_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} \mathcal{F}^\Sigma \\ \mathcal{G}_\Sigma \end{pmatrix}. \tag{2.3}$$

The new field strengths $\mathcal{G}_\Lambda$ can be written in the form (2.2) with a new lagrangian, provided that the matrix in (2.3) constitutes an element of the group $\mathrm{Sp}(2n, \mathbb{R})$. Obviously these transformations are generalizations of the duality transformations known from Maxwell theory, which rotate the electric and magnetic fields and inductions (for a review of electric/magnetic duality, see [10]).

We will employ an $\mathrm{Sp}(2n, \mathbb{R})$ covariant notation for the $2n$-dimensional symplectic indices $M, N, \ldots$, such that $Z^M = (Z^\Lambda, Z_\Lambda)$. Likewise we use vectors with lower indices according to $Y_M = (Y_\Lambda, Y^\Lambda)$, transforming according to the conjugate representation so that $Z^M Y_M$ is invariant. Our conventions are such that the $\mathrm{Sp}(2n, \mathbb{R})$ invariant skew-symmetric tensor Ω_{MN} takes the form,

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \tag{2.4}$$

The conjugate matrix Ω^{MN} is defined by $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$.

In the following we will be dealing with lagrangians that are at most quadratic in the field strengths, although our methods can also be applied to more complicated lagrangians.

In addition the lagrangian may depend on other fields. Let us consider a generalization of the kinetic term depending on a (possibly field-dependent) symmetric tensor $\mathcal{N}_{\Lambda\Sigma}$,

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4}\mathrm{i}\Big\{\mathcal{N}_{\Lambda\Sigma}\,\mathcal{F}^{+\,\Lambda}_{\mu\nu}\,\mathcal{F}^{+\mu\nu\Sigma} - \bar{\mathcal{N}}_{\Lambda\Sigma}\,\mathcal{F}^{-\,\Lambda}_{\mu\nu}\,\mathcal{F}^{-\mu\nu\Sigma}\Big\}\\ &= \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\,\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}_{\rho\sigma}{}^{\Sigma}\,,\end{aligned}\tag{2.5}$$

where the $\mathcal{F}^{\pm}_{\mu\nu}$ are complex selfdual combinations with eigenvalue $\mp\mathrm{i}$ normalized such that $\mathcal{F}_{\mu\nu} = \mathcal{F}^{+}_{\mu\nu} + \mathcal{F}^{-}_{\mu\nu}$; $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts of $\mathcal{N}$ and play the role of generalized theta angles and coupling constants, respectively.

Upon an electric/magnetic duality transformation (2.3) one finds an alternative lagrangian of the same form but with a different expression for $\mathcal{N}_{\Lambda\Sigma}$,

$$\mathcal{N}_{\Lambda\Sigma} \longrightarrow (V\mathcal{N} + W)_{\Lambda\Gamma}\,\left[(U + Z\mathcal{N})^{-1}\right]^{\Gamma}{}_{\Sigma}\,.\tag{2.6}$$

This result follows from requiring consistency between (2.2) and (2.3). The symmetry of the new $\mathcal{N}$ is ensured by the fact that (2.3) belongs to $\mathrm{Sp}(2n,\mathbb{R})$. Two lagrangians with tensors $\mathcal{N}_{\Lambda\Sigma}$ related via (2.6), are equivalent by electric/magnetic duality. However, if the tensor $\mathcal{N}_{\Lambda\Sigma}$ depends on fields whose transformations induce precisely a change of $\mathcal{N}_{\Lambda\Sigma}$ of the form (2.6), then we may be dealing with an invariance of the combined field equations and Bianchi identities [11]. Of course, this invariance is only realized provided that also the field equations associated with fields other than the gauge fields, will respect the invariance. For instance, the corresponding transformations of the scalar fields should leave the scalar kinetic term invariant. Since this term takes the form of a non-linear sigma model, the transformations must constitute isometries of the target space manifold.

The above transformations can be realized for more general lagrangians. In particular we can introduce a moment coupling of the form,

$$\mathcal{L}_{\mathrm{m}} = \mathcal{F}^{+\Lambda}_{\mu\nu}\,\mathcal{O}^{+\mu\nu}_{\Lambda} + \mathcal{F}^{-\Lambda}_{\mu\nu}\,\mathcal{O}^{-\mu\nu}_{\Lambda}\,,\tag{2.7}$$

where $\mathcal{O}^{\pm}_{\mu\nu\,\Lambda}$ depends on matter fields and is usually bilinear in spinor fields. Equivalent lagrangians are now defined in terms of tensors $\mathcal{N}$ related according to (2.6) and tensors $\mathcal{O}^{\pm}$ related according to

$$\mathcal{O}^{+}_{\mu\nu\,\Lambda} \longrightarrow \mathcal{O}^{+}_{\mu\nu\,\Sigma}\,[(U + Z\mathcal{N})^{-1}]^{\Sigma}{}_{\Lambda}\,,\tag{2.8}$$

and likewise for $\mathcal{O}^{-}_{\mu\nu\,\Lambda}$, but with $\mathcal{N}$ replaced by its complex conjugate. In order to have an invariance, the transformations (2.8) should be induced by the transformations of the fields on which the tensors $\mathcal{O}^{\pm}_{\mu\nu\,\Lambda}$ depend. In the presence of the moment coupling (2.7), it is advantageous to also include the following matter term into the lagrangian [10],

$$\mathcal{L}'{}_{\mathrm{m}} = \frac{1}{2}[\mathcal{I}^{-1}]^{\Lambda\Sigma}\,\mathcal{O}_{\mu\nu\,\Lambda}\,\mathcal{O}^{\mu\nu}{}_{\Sigma}\,,\tag{2.9}$$

with $\mathcal{O}_{\Lambda} = \mathcal{O}^{+}_{\Lambda} + \mathcal{O}^{-}_{\Lambda}$. While this term is itself not invariant, it ensures that the unspecified remaining terms in the total lagrangian will be separately invariant.

In this paper we will be dealing with the full group of invariances which we denote by G. According to the above, the invariance transformations that act on the vector fields should always comprise a subgroup of the electric/magnetic duality group. This implies that a $2n$-dimensional representation of G should exist with generators $(t_\alpha)_M{}^N$, where the indices α label the generators, satisfying

$$(t_\alpha)_{[M}{}^P\,\Omega_{N]P} = 0\,. \tag{2.10}$$

On a $2n$-dimensional symplectic vector Z_M, such a transformation takes the form $\delta Z_M = \Lambda^\alpha\,(t_\alpha)_M{}^N\,Z_N$, where the matrix decomposes according to

$$\Lambda^\alpha\,(t_\alpha)_M{}^N = \begin{pmatrix} b_\Lambda{}^\Sigma & c_{\Lambda\Sigma} \\ d^{\Lambda\Sigma} & -(b^{\rm T})^\Lambda{}_\Sigma \end{pmatrix}\,, \tag{2.11}$$

with $c_{\Lambda\Sigma} = c_{\Sigma\Lambda}$ and $d^{\Lambda\Sigma} = d^{\Sigma\Lambda}$. The matrices b, c, d comprise at most $n(2n{+}1)$ independent parameters, which is consistent with the fact that we are dealing with a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. Observe that the above conventions are such that the infinitesimal form of the matrix in (2.3) reads as follows, $U \approx 1 - b^{\rm T}$, $V \approx 1 + b$, $W \approx -c$, $Z \approx -d$.

For continuous invariances there is a simple way to determine the explicit form of the submatrices b, c and d. Namely one sandwiches (2.11) with the symplectic vectors $(\mathcal{F}_{\mu\nu}{}^\Lambda, \mathcal{G}_{\mu\nu\,\Lambda})$ and its dual $(\mathcal{G}_{\rho\sigma\,\Sigma}, -\mathcal{F}_{\rho\sigma}{}^\Sigma)$ and contracts over $\varepsilon^{\mu\nu\rho\sigma}$. The resulting expression must vanish (see [10, eq. (12)]), which is a rather stringent condition on the generators.

We caution the reader that at this point the invariance applies to the combined equations of motion and the Bianchi identities, while the lagrangian is in general *not* invariant. In principle there is nothing wrong with this and supergravity theories have provided many examples of theories where this situation is realized.

The dual field strengths (2.2) derived from the combined lagrangian (2.5), (2.7) read

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathcal{R}_{\Lambda\Gamma}\mathcal{F}_{\mu\nu}{}^\Gamma - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{I}_{\Lambda\Gamma}\,\mathcal{F}^{\rho\sigma\,\Gamma} - \varepsilon_{\mu\nu\rho\sigma}\,\mathcal{O}^{\rho\sigma}{}_\Lambda\,. \tag{2.12}$$

The relation (2.12) is consistent with all the transformation rules given previously. So far we only introduced electric gauge fields $A_\mu{}^\Lambda$, but at this stage one can also introduce their magnetic duals $A_{\mu\,\Lambda}$ associated with these dual field strengths $\mathcal{G}_{\mu\nu\,\Lambda}$, by writing $\mathcal{G}_{\mu\nu\,\Lambda} \equiv 2\,\partial_{[\mu}A_{\nu]\Lambda}$. The invariance group G mixes the two types of field strengths, as follows from (2.3). Therefore the generators should be viewed as generalized charges that contain both electric and magnetic components. Switching on a gauge coupling constant may thus require both electric and magnetic vector potentials.

3. Gauging with electric and magnetic potentials

We now introduce gauge couplings into the lagrangian without restricting ourselves to only electric charges. Hence we introduce gauge fields $A_\mu{}^M$ which decompose into electric gauge fields $A_\mu{}^\Lambda$ and magnetic gauge fields $A_{\mu\,\Lambda}$. Of course, usually only a subset of these fields will be involved in the gauging. Introducing magnetic gauge fields could lead to additional propagating degrees of freedom. We will discuss in due course how to avoid this.

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The gauge group must be embedded into the rigid invariance group. This is done by means of an embedding tensor $\Theta_M{}^\alpha$ which determines the decomposition of the gauge group generators X_M into the generators associated with the rigid invariance group G,

$$X_M = \Theta_M{}^\alpha\, t_\alpha\,. \tag{3.1}$$

Not all the gauge fields have to be involved in the gauging, so generically the embedding tensor projects out certain combinations of gauge fields; the rank of the tensor determines the dimension of the gauge group, up to central extensions. Decomposing the embedding tensor as $\Theta_M{}^\alpha = (\Theta_\Lambda{}^\alpha, \Theta^{\Lambda\,\alpha})$, covariant derivatives take the form,

$$D_\mu \equiv \partial_\mu - g A_\mu{}^M X_M = \partial_\mu - g A_\mu{}^\Lambda \Theta_\Lambda{}^\alpha\, t_\alpha - g A_{\mu\,\Lambda} \Theta^{\Lambda\,\alpha}\, t_\alpha\,. \tag{3.2}$$

As stressed in section 1, the embedding tensor is treated as a spurionic object, which can then be assigned to a (not necessarily irreducible) representation of the rigid invariance group G.

From our experience with supergravity, we know that a number of (G-covariant) constraints must be imposed on the embedding tensor. We introduce two such constraints quadratic in the embedding tensor,

$$f_{\alpha\beta}{}^\gamma\, \Theta_M{}^\alpha\, \Theta_N{}^\beta + (t_\alpha)_N{}^P\, \Theta_M{}^\alpha \Theta_P{}^\gamma = 0\,, \tag{3.3}$$

$$\Omega^{MN}\, \Theta_M{}^\alpha \Theta_N{}^\beta \;=\; 0 \;\Longleftrightarrow\; \Theta^{\Lambda\,[\alpha}\Theta_\Lambda{}^{\beta]} = 0\,, \tag{3.4}$$

where the $f_{\alpha\beta}{}^\gamma$ are the structure constants associated with the group G. The first constraint is required by the closure of the gauge group generators. Indeed, from (3.3) it follows that the gauge algebra generators close according to

$$[X_M, X_N] = -X_{MN}{}^P\, X_P\,, \tag{3.5}$$

where the structure constants of the gauge group coincide with $X_{MN}{}^P \equiv \Theta_M{}^\alpha\, (t_\alpha)_N{}^P$ up to terms that vanish upon contraction with the embedding tensor $\Theta_P{}^\alpha$. We recall that the $X_{MN}{}^P$ generate a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ in the $(2n)$-dimensional representation, so that $X_{M\Lambda}{}^\Sigma = -X_M{}^\Sigma{}_\Lambda$, $X_{M\Lambda\Sigma} = X_{M\Sigma\Lambda}$ and $X_M{}^{\Lambda\Sigma} = X_M{}^{\Sigma\Lambda}$. Note that (3.3) also establishes the gauge invariance of the embedding tensor. The second quadratic constraint (3.4) implies that the charges are mutually local, so that an electric/magnetic duality exists that will convert all the charges to electric ones.

In addition, we impose the following (G-covariant) linear constraint on $\Theta_M{}^\alpha$,

$$X_{(MN}{}^Q\, \Omega_{P)Q} = 0 \Longrightarrow \begin{cases} X^{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2X^{(\Gamma\Lambda)}{}_\Sigma = X_\Sigma{}^{\Lambda\Gamma}\,, \\ X_{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2X_{(\Gamma\Lambda)}{}^\Sigma = X^\Sigma{}_{\Lambda\Gamma}\,, \end{cases} \tag{3.6}$$

which implies that we suppress a number of independent irreducible representations that are generically contained in the embedding tensor.

Obviously one can impose additional constraints on the embedding tensor, but the above set is probably the minimal one. The constraints (3.3) and (3.6) are known from maximal $N = 8$ supergravity [4, 12, 13], where (3.6) is required by the supersymmetry of the action. These two constraints imply the validity of the third one (3.4). The relation between the two quadratic constraints turns out to be a more generic feature, as we can see by symmetrizing the constraint (3.3) in (MN) and by using the linear constraint (3.6) and (2.10). This leads to

$$\Omega^{MN}\,\Theta_M{}^\alpha\Theta_N{}^\beta\,(t_\beta)_P{}^Q = 0\,, \tag{3.7}$$

which shows that for nonvanishing $(t_\beta)_P{}^Q$ the second quadratic constraint (3.4) is in fact a consequence of the other constraints just as for the $N = 8$ theory. Only for those generators t_α that have a trivial action on the vector fields, does (3.4) represent an independent constraint. This happens only when the symmetry group of the ungauged theory factorizes into a product of several groups. We will encounter this situation later in section 5.

As a further consequence of (3.6) one finds that

$$X_{(MN)}{}^P = Z^{P,\alpha}\,d_{\alpha\,MN}\,, \tag{3.8}$$

with

$$\begin{aligned} d_{\alpha\,MN} &\equiv (t_\alpha)_M{}^P\,\Omega_{NP}\,,\\ Z^{M,\alpha} &\equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^\alpha \quad\Longrightarrow\quad \begin{cases} Z^{\Lambda,\alpha} = \dfrac{1}{2}\Theta^{\Lambda\alpha}\,,\\ Z_\Lambda{}^{,\alpha} = -\dfrac{1}{2}\Theta_\Lambda{}^\alpha\,. \end{cases} \end{aligned} \tag{3.9}$$

The tensor $d_{\alpha\,MN}$ defines a G-invariant tensor symmetric in (MN). The gauge invariant tensor $Z^{M,\alpha}$ will serve as a projector on the tensor fields to be introduced in the following [2]. By virtue of the constraint (3.4), we have

$$Z^{M,\alpha}\,\Theta_M{}^\beta = 0\,. \tag{3.10}$$

Let us return to the closure relation (3.5). Although the left-hand side is antisymmetric in M and N, this does not imply that $X_{MN}{}^P$ is antisymmetric as well, but only that its symmetric part vanishes upon contraction with the embedding tensor. Indeed, this is reflected by (3.8) and (3.10). Consequently, the Jacobi identity holds only modulo terms that vanish upon contraction with the embedding tensor, as is shown by

$$X_{[MN]}{}^P\,X_{[QP]}{}^R + X_{[QM]}{}^P\,X_{[NP]}{}^R + X_{[NQ]}{}^P\,X_{[MP]}{}^R = -Z^{R,\alpha}\,d_{\alpha\,P[Q}\,X_{MN]}{}^P\,. \tag{3.11}$$

To compensate for this lack of closure and, at the same time, to avoid unwanted degrees of freedom, we introduce an extra gauge invariance for the gauge fields, in addition to the usual nonabelian gauge transformations,

$$\delta A_\mu{}^M = D_\mu\Lambda^M - g\,Z^{M,\alpha}\,\Xi_{\mu\,\alpha}\,, \tag{3.12}$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $D_\mu\Lambda^M = \partial_\mu\Lambda^M + g\,X_{PQ}{}^M\,A_\mu{}^P\Lambda^Q$. The transformations proportional to $\Xi_{\mu\,\alpha}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $X_{MN}{}^P$ where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor).[1] Because the Jacobi identity is not satisfied and because of the extra gauge transformations, the usual field strength, which follows from the Ricci identity, $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M X_M$,

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{[NP]}{}^M\, A_\mu{}^N A_\nu{}^P\,, \tag{3.13}$$

is not fully covariant.[2] Therefore we define a modified field strength,

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\, Z^{M,\alpha}\, B_{\mu\nu\,\alpha}\,, \tag{3.14}$$

where we introduce the tensor fields $B_{\mu\nu\,\alpha}$, whose transformation rules will be defined such that the $\mathcal{H}_{\mu\nu}{}^M$ transform covariantly,

$$\delta\mathcal{H}_{\mu\nu}{}^M = -g\, X_{PN}{}^M\, \Lambda^P \mathcal{H}_{\mu\nu}{}^N\,. \tag{3.15}$$

This leads to the following result for the transformation rule of $B_{\mu\nu\,\alpha}$,[3]

$$\delta B_{\mu\nu\,\alpha} = 2\, D_{[\mu}\Xi_{\nu]\alpha} + 2\, d_{\alpha\,MN} A_{[\mu}{}^M \delta A_{\nu]}{}^N - 2\, d_{\alpha\,MN}\Lambda^M \mathcal{H}_{\mu\nu}{}^N\,, \tag{3.16}$$

up to terms that vanish under contraction with $Z^{M,\alpha}$. As it turns out, we do not need these contributions as variations of the tensor field in the final lagrangian will always be multiplied by $Z^{\Lambda,\alpha}$. The relevant variation is therefore,

$$\Theta^{\Lambda\alpha}\delta B_{\mu\nu\,\alpha} = 2\,\Theta^{\Lambda\alpha}\left[D_{[\mu}\Xi_{\nu]\alpha} + d_{\alpha\,MN} A_{[\mu}{}^M \delta A_{\nu]}{}^N\right] - 2\,\Lambda^M \left[X^\Lambda{}_{M\Sigma}\,\mathcal{H}_{\mu\nu}{}^\Sigma - X^\Lambda{}_M{}^\Sigma\,\mathcal{H}_{\mu\nu\Sigma}\right], \tag{3.17}$$

where we made use of (3.9).

In passing we note that the covariant field strength of the tensor fields is known and given by [2],

$$\mathcal{H}^{(3)}_{\mu\nu\rho\,\alpha} \equiv 3\, D_{[\mu} B_{\nu\rho]\,\alpha} + 6\, d_{\alpha\,MN}\, A_{[\mu}{}^M (\partial_\nu A_{\rho]}{}^N + \frac{1}{3} g X_{[RS]}{}^N A_\nu{}^R A_{\rho]}{}^S)\,, \tag{3.18}$$

up to terms that vanish when contracted with $Z^{M,\alpha}$. The vector and tensor field strengths satisfy the generalized Bianchi identities (the tensor identity holds upon contraction with $Z^{M,\alpha}$),

$$Z^{M,\alpha} D_{[\mu}\mathcal{H}^{(3)}_{\nu\rho\sigma]\alpha} = 3g\, X_{PQ}{}^M\, \mathcal{H}_{[\mu\nu}{}^P\, \mathcal{H}_{\rho\sigma]}{}^Q\,, \tag{3.19}$$

$$D_{[\mu}\mathcal{H}_{\nu\rho]}{}^M = -\frac{1}{3} g\, Z^{M,\alpha}\, \mathcal{H}^{(3)}_{\mu\nu\rho\,\alpha}\,, \tag{3.20}$$

with the covariant derivatives $D\mathcal{H}^M = \partial\mathcal{H}^M + g\, X_{PQ}{}^M A^P\, \mathcal{H}^Q$ and $D\mathcal{H}_\alpha = \partial\mathcal{H}_\alpha + g\,\Theta_M{}^\gamma f_{\gamma\alpha}{}^\beta\, A^M \mathcal{H}_\beta$.

[1]Here we modified the gauge field transformation rules as compared to earlier publications [1, 2], which simply amounts to a redefinition, $\Xi_{\mu\,\alpha} \to \Xi_{\mu\,\alpha} - d_{\alpha PQ} A_\mu{}^P \Lambda^Q$. This modification will lead to certain simplifications later on.

[2]Observe that the covariant derivative is invariant under the tensor gauge transformations, so that the field strengths contracted with X_M are in fact covariant.

[3]This result is taken from [2], but it takes a different form due to the redefinition of $\Xi_{\mu\,\alpha}$, discussed in footnote 1.

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The next steps are rather obvious. Namely one covariantizes the combined lagrangian (2.5), (2.7) and adds a topological term that involves the tensor and vector fields. However, in four space-time dimensions this will not directly lead to the correct solution and further modifications will be required. This is related to the fact that the rigid invariance G was not necessarily an invariance of the initial lagrangian, but of the combined equations of motion and the Bianchi identities. In this section we will therefore carry out these first steps and exhibit the problematic features of this intermediate result. In addition we will show that the purely electric gaugings do not suffer from any of these problems. In section 4 we will then introduce the complete gauge invariant lagrangian and transformation rules. In all of this the embedding tensor constraints play a crucial role.

Covariantizing the combined lagrangian (2.5), (2.7) leads to

$$\mathcal{L}_0 + \mathcal{L}_{\rm m} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\,\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{\Sigma} + \mathcal{H}_{\mu\nu}{}^{\Lambda}\,\mathcal{O}^{\mu\nu}{}_{\Lambda}\,. \tag{3.21}$$

However, this lagrangian is not invariant for the same reason as the original lagrangian was not invariant. To see this one makes use of the infinitesimal gauge transformations, which for $\mathcal{N}_{\Lambda\Sigma}$ and $\mathcal{O}^{+}_{\mu\nu\,\Lambda}$ take the form,

$$\begin{aligned} \delta\mathcal{N}_{\Lambda\Sigma} &= g\,\Lambda^M\left[-X_{M\Lambda\Sigma} + 2\,X_{M(\Lambda}{}^{\Gamma}\mathcal{N}_{\Sigma)\Gamma} + \mathcal{N}_{\Lambda\Gamma}\,X_M{}^{\Gamma\Xi}\,\mathcal{N}_{\Xi\Sigma}\right], \\ \delta O^{+}_{\mu\nu\,\Lambda} &= g\,\Lambda^M\,O^{+}_{\mu\nu\,\Sigma}\left[X_{M\Lambda}{}^{\Sigma} + X_M{}^{\Sigma\Gamma}\,\mathcal{N}_{\Gamma\Lambda}\right]. \end{aligned} \tag{3.22}$$

Using the variations (3.15) and (3.22), one derives

$$\begin{aligned} \delta(\mathcal{L}_0 + \mathcal{L}_{\rm m} + \mathcal{L}'{}_{\rm m}) = &-\frac{1}{8}g\,\Lambda^M X_{M\Lambda\Sigma}\,\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{\Sigma}\,\varepsilon^{\mu\nu\rho\sigma} + \\ &+\frac{1}{8}g\,\Lambda^M X_M{}^{\Lambda\Sigma}\,\mathcal{G}_{\mu\nu\Lambda}\,\mathcal{G}_{\rho\sigma\Sigma}\,\varepsilon^{\mu\nu\rho\sigma} - \\ &-\frac{1}{4}g\,\Lambda^M X_M{}^{\Lambda\Sigma}\,\mathcal{G}_{\mu\nu\Lambda}\,\mathcal{H}_{\rho\sigma\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\,, \end{aligned} \tag{3.23}$$

where

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathcal{R}_{\Lambda\Gamma}\mathcal{H}_{\mu\nu}{}^{\Gamma} - \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\,\mathcal{I}_{\Lambda\Gamma}\,\mathcal{H}^{\rho\lambda\,\Gamma} - \varepsilon_{\mu\nu\rho\lambda}\,\mathcal{O}^{\rho\lambda}{}_{\Lambda}\,, \tag{3.24}$$

is a covariant version of (2.12). Furthermore we note that the variation with respect to $B_{\mu\nu\alpha}$ leads also to the tensor $\mathcal{G}_{\mu\nu\,\Lambda}$,

$$\delta(\mathcal{L}_0 + \mathcal{L}_{\rm m} + \mathcal{L}'{}_{\rm m}) = \frac{1}{8}g\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{G}_{\mu\nu\Lambda}\,\Theta^{\Lambda\alpha}\,\delta B_{\mu\nu\,\alpha}\,. \tag{3.25}$$

On the basis of the results found for maximal supergravity in five space-time dimensions [1] and the more general considerations presented in [2], we introduce a topological term, $g\,\Theta^{\Lambda\,\alpha}\,\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu\,\alpha}\,\partial_\rho A_{\sigma\,\Lambda}$, which only contains the magnetic potentials so that the tensor field will decouple for purely electric gaugings (the need for this will be explained below). The term $\partial_{[\rho}A_{\sigma]\Lambda}$ constitutes the first term of the field strength $\mathcal{H}_{\rho\sigma\Lambda}$, which suggests that its completion will at least involve the terms,

$$\mathcal{L}_{{\rm top},B} = -\frac{1}{8}g\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{\Lambda\alpha}\,B_{\mu\nu\,\alpha}\left(2\,\partial_\rho A_{\sigma\,\Lambda} + gX_{MN\,\Lambda}\,A_\rho{}^M A_\sigma{}^N - \frac{1}{4}g\,\Theta_\Lambda{}^\beta B_{\rho\sigma\,\beta}\right), \tag{3.26}$$

so that its variation with respect to $\delta B_{\mu\nu\alpha}$ is just proportional to $\mathcal{H}_{\mu\nu\Lambda}$,

$$\delta\mathcal{L}_{\text{top},B} = -\frac{1}{8}g\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{\mu\nu\Lambda}\,\Theta^{\Lambda\alpha}\,\delta B_{\rho\sigma\,\alpha}\;. \tag{3.27}$$

Note that the tensor $\Theta^{\Lambda\alpha}\,\Theta_\Lambda{}^\beta$ that multiplies the term quadratic in $B_{\mu\nu\Lambda}$ is symmetric in (α,β), by virtue of the constraint (3.4). General variations of (3.26) can be written as follows,

$$\delta\mathcal{L}_{\text{top},B} = -\frac{1}{8}g\,\varepsilon^{\mu\nu\rho\sigma}\left[\mathcal{F}_{\mu\nu\Lambda}\,\Theta^{\Lambda\alpha}\,\delta B_{\rho\sigma\,\alpha} + \delta\mathcal{H}_{\mu\nu\Lambda}\;\Theta^{\Lambda\alpha}\;B_{\rho\sigma\,\alpha}\right]\;. \tag{3.28}$$

Substituting the various variations, one finds,

$$\begin{aligned}\delta\mathcal{L}_{\text{top},B} = {}& \frac{1}{8}g\,\Lambda^M X_{M\Lambda\Sigma}\Big[\mathcal{H}_{\mu\nu}{}^\Lambda\mathcal{H}_{\rho\sigma}{}^\Sigma - \mathcal{F}_{\mu\nu}{}^\Lambda\mathcal{F}_{\rho\sigma}{}^\Sigma\Big]\,\varepsilon^{\mu\nu\rho\sigma} + \\ & + \frac{1}{8}g\,\Lambda^M X_M{}^{\Lambda\Sigma}\Big[\mathcal{H}_{\mu\nu\Lambda}\mathcal{H}_{\rho\sigma\Sigma} - \frac{1}{2}\mathcal{F}_{\mu\nu\Lambda}\mathcal{F}_{\rho\sigma\Sigma}\Big]\varepsilon^{\mu\nu\rho\sigma} + \\ & + \frac{1}{8}g\,\Lambda^M X^\Lambda{}_{M\Sigma}\,\mathcal{F}_{\mu\nu\Lambda}\,\mathcal{F}_{\rho\sigma}{}^\Sigma\,\varepsilon^{\mu\nu\rho\sigma}\,,\end{aligned} \tag{3.29}$$

up to terms of order g^2 that contain noncovariant terms which depend explicitly on $A_\mu{}^M$. The constraints on the embedding tensor are crucial for deriving the above results.

Clearly at this stage the combined lagrangian is not invariant as the tensors $\mathcal{G}_{\mu\nu\Lambda}$ and $\mathcal{H}_{\mu\nu\Lambda}$ are unrelated, although we note that the terms quadratic in $\mathcal{H}_{\mu\nu}{}^M$ cancel when $\mathcal{G}_{\mu\nu\Lambda}$ and $\mathcal{H}_{\mu\nu\Lambda}$ are identified. This observation will be relevant later on. However, one is then still left with the terms quadratic in $\mathcal{F}_{\mu\nu}{}^M$. We will exhibit in the next section how these variations are cancelled. To pave the way and to verify the consistency of the construction up to this point, let us briefly consider purely electric gaugings, to appreciate the role of the constraints and to establish that our formalism will remain in the more conventional setting. For electric gaugings, $\Theta^{\Lambda\alpha}=0$, so that the generators $X^\Lambda=0$. In that case the constraint (3.6) reduces to

$$X_\Sigma{}^{\Lambda\Gamma} = 0\,,\qquad X_{(\Lambda\Sigma)}{}^\Gamma = 0\,,\qquad X_{(\Lambda\Sigma\Gamma)} = 0\,. \tag{3.30}$$

The remaining gauge generators, $X_{[\Lambda\Sigma]}{}^\Gamma = X_\Lambda{}^\Gamma{}_\Sigma$ and $X_{\Lambda\Sigma\Gamma}$ satisfy the Jacobi identity, because the right-hand side of (3.11) vanishes. Hence the generators have a block-triangular form. The modified field strength (3.14) for the electric vector fields reduces to an ordinary nonabelian field strength

$$\mathcal{H}_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu}A_{\nu]}{}^\Lambda + g\,X_{\Sigma\Gamma}{}^\Lambda\,A_\mu{}^\Sigma A_\nu{}^\Gamma\,, \tag{3.31}$$

which contains neither magnetic vector fields nor tensor fields. The variation of the lagrangian follows from (3.23), where only the first term contributes. It was observed long ago in [6] that this variation can be cancelled by introducing the following Chern-Simons-like term to the action,

$$\mathcal{L}_{\text{top,electric}} = -\frac{1}{3}g\,\varepsilon^{\mu\nu\rho\sigma}X_{\Omega\Xi\Sigma}\,A_\mu{}^\Omega A_\nu{}^\Xi\left(\partial_\rho A_\sigma{}^\Sigma + \frac{3}{8}g\,X_{\Lambda\Gamma}{}^\Sigma\,A_\rho{}^\Lambda A_\sigma{}^\Gamma\right), \tag{3.32}$$

provided $X_{(\Lambda\Sigma\Gamma)}=0$, which is precisely the last equation of (3.30). This extends possible gauge transformations to those with triangular embedding into the symplectic group (2.3), the so-called Peccei-Quinn transformations.

4. Tensor fields and the topological term

In this section we demonstrate that a general gauge invariant lagrangian exists with both electric and magnetic vector potentials. This lagrangian is an extension of the lagrangian discussed in the previous section, by Chern-Simons-like terms such as (3.32). The only restriction will be that the embedding tensor $\Theta_M{}^\alpha$ is subject to the constraints (3.3), (3.4) and (3.6).

The first observation is that the variations of the total lagrangian bilinear in $\mathcal{H}_{\rho\sigma\Sigma}$ and $\mathcal{G}_{\mu\nu\Lambda}$ combine into $X_M{}^{\Lambda\Sigma}(\mathcal{H}_{\mu\nu\Lambda}-\mathcal{G}_{\mu\nu\Lambda})(\mathcal{H}_{\rho\sigma\Sigma}-\mathcal{G}_{\rho\sigma\Sigma})$, which, according to (3.25) and (3.27), can be removed by assigning an additional variation to $\delta B_{\mu\nu\alpha}$ proportional to $(\mathcal{H}_{\mu\nu\Lambda}-\mathcal{G}_{\mu\nu\Lambda})$. With this variation the modified transformation rule for $B_{\mu\nu\alpha}$ reads,

$$\Theta^{\Lambda\alpha}\delta B_{\mu\nu\,\alpha} = 2\,\Theta^{\Lambda\alpha}\Big[D_{[\mu}\Xi_{\nu]\alpha} + d_{\alpha\,MN}A_{[\mu}{}^M\delta A_{\nu]}{}^N\Big] - 2\,\Lambda^M\Big[X^\Lambda{}_{M\Sigma}\,\mathcal{H}_{\mu\nu}{}^\Sigma - X^\Lambda{}_M{}^\Sigma\,\mathcal{G}_{\mu\nu\Sigma}\Big]\,. \tag{4.1}$$

Apart from this modification the remaining transformation rules are left unchanged, but one should be aware that (3.15) receives corrections induced by the modifications in (4.1). With these transformation rules, the variation of the total lagrangian takes the form

$$\begin{aligned}\delta(\mathcal{L}_0+\mathcal{L}_{\rm m}+\mathcal{L}'_{\rm m}+\mathcal{L}_{{\rm top},B}) = {}& -\frac{1}{8}g\,\Lambda^M X_{M\Lambda\Sigma}\,\mathcal{F}_{\mu\nu}{}^\Lambda\mathcal{F}_{\rho\sigma}{}^\Sigma\,\varepsilon^{\mu\nu\rho\sigma} - \\ & -\frac{1}{16}g\,\Lambda^M X_M{}^{\Lambda\Sigma}\,\mathcal{F}_{\mu\nu\,\Lambda}\mathcal{F}_{\rho\sigma\,\Sigma}\,\varepsilon^{\mu\nu\rho\sigma} + \\ & +\frac{1}{8}g\,\Lambda^M X^\Lambda{}_{M\Sigma}\,\mathcal{F}_{\mu\nu\,\Lambda}\mathcal{F}_{\rho\sigma}{}^\Sigma\,\varepsilon^{\mu\nu\rho\sigma} + \\ & +\cdots\,,\end{aligned} \tag{4.2}$$

where the dots denote noncovariant terms of order g^2 which depend explicitly on $A_\mu{}^M$. In order to finally cancel this variation, the topological lagrangian (3.26) must be extended to

$$\begin{aligned}\mathcal{L}_{\rm top} = {}& -\frac{1}{8}g\,\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{\Lambda\alpha}\,B_{\mu\nu\,\alpha}\Big(2\,\partial_\rho A_{\sigma\,\Lambda} + gX_{MN\,\Lambda}\,A_\rho{}^M A_\sigma{}^N - \frac{1}{4}g\Theta_\Lambda{}^\beta B_{\rho\sigma\,\beta}\Big) - \\ & -\frac{1}{3}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN\,\Lambda}\,A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_\sigma{}^\Lambda + \frac{1}{4}gX_{PQ}{}^\Lambda A_\rho{}^P A_\sigma{}^Q\Big) - \\ & -\frac{1}{6}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN}{}^\Lambda\,A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4}gX_{PQ\Lambda}A_\rho{}^P A_\sigma{}^Q\Big)\,.\end{aligned} \tag{4.3}$$

Straightforward but tedious computation then shows that the variation of the extra terms precisely cancel the contributions (4.2) such that the sum

$$\mathcal{L}_{\rm VT} = \mathcal{L}_0 + \mathcal{L}_{\rm m} + \mathcal{L}'_{\rm m} + \mathcal{L}_{\rm top}\,, \tag{4.4}$$

is invariant under both vector and tensor gauge transformations up to total derivatives. The constraints (3.3), (3.4), and (3.6) are crucial in the derivation of this result. The topological term (4.3) contains the first-order term (3.26) for the magnetic vector fields A_Λ and the tensor fields B_α and the Chern-Simons-like term (3.32). Indeed, for an electric gauging ($\Theta^{\Lambda\alpha}=0$) the Chern-Simons-like terms in (4.3) reduce to (3.32), while for a purely magnetic gauging ($\Theta_\Lambda{}^\alpha = 0$), they take the form

$$\begin{aligned}\mathcal{L}_{\text{top, magnetic}} = & -\frac{1}{8} g\, \varepsilon^{\mu\nu\rho\sigma}\, \Theta^{\Lambda\alpha}\, B_{\mu\nu\,\alpha}\Big(2\,\partial_\rho A_{\sigma\Lambda} + g\, X^{\Sigma}{}_{M\Lambda}\, A_{\rho\Sigma} A_\sigma{}^M\Big) - \\ & -\frac{1}{6} g\, \varepsilon^{\mu\nu\rho\sigma} X^{\Omega\Xi\Sigma}\, A_{\mu\,\Omega} A_{\nu\,\Xi}\Big(\partial_\rho A_{\sigma\Sigma} + \frac{3}{8} g\, X^{\Lambda\Gamma}{}_\Sigma\, A_{\rho\Lambda} A_{\sigma\Gamma}\Big) - \\ & -\frac{1}{4} g\, \varepsilon^{\mu\nu\rho\sigma} X^{\Omega\Xi}{}_\Sigma\, A_{\mu\,\Omega} A_{\nu\,\Xi}\Big(\partial_\rho A_\sigma{}^\Sigma + \frac{1}{4} g\, X^{\Lambda\Gamma\Sigma}\, A_{\rho\Lambda} A_{\sigma\Gamma}\Big)\,. \end{aligned} \tag{4.5}$$

Summarizing, we have shown that the total lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{VT}} = & \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\, \mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{H}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\, \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{H}_{\rho\sigma}{}^\Sigma + \mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{O}^{\mu\nu}{}_\Lambda + \frac{1}{2}[\mathcal{I}^{-1}]^{\Lambda\Sigma}\, \mathcal{O}_{\mu\nu\,\Lambda} \mathcal{O}^{\mu\nu}{}_\Sigma - \\ & -\frac{1}{8} g\, \varepsilon^{\mu\nu\rho\sigma}\, \Theta^{\Lambda\alpha}\, B_{\mu\nu\,\alpha}\,\Big(2\,\partial_\rho A_{\sigma\,\Lambda} + g X_{MN\,\Lambda}\, A_\rho{}^M A_\sigma{}^N - \frac{1}{4} g \Theta_\Lambda{}^\beta B_{\rho\sigma\,\beta}\Big) - \\ & -\frac{1}{3} g\, \varepsilon^{\mu\nu\rho\sigma} X_{MN\,\Lambda}\, A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_\sigma{}^\Lambda + \frac{1}{4} g X_{PQ}{}^\Lambda A_\rho{}^P A_\sigma{}^Q\Big) - \\ & -\frac{1}{6} g\, \varepsilon^{\mu\nu\rho\sigma} X_{MN}{}^\Lambda\, A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4} g X_{PQ\Lambda} A_\rho{}^P A_\sigma{}^Q\Big)\,, \end{aligned} \tag{4.6}$$

is invariant under the vector and tensor gauge transformations (3.12), (3.22) and (4.1). It provides a unified description of electric and magnetic vector fields as well as of tensor fields which encompasses all possible gaugings. The gauge group is characterized by the embedding tensor $\Theta_M{}^\alpha$ subject to the constraints (3.3), (3.4), and (3.6). Apart from these constraints the embedding of the gauge group into the symplectic group (2.3) is arbitrary. Due to the presence of both electric and magnetic vector fields the gauge group is no longer restricted to diagonal or triangular embeddings. The gaugings are thus not necessarily restricted to subgroups of the invariance group of the initial ungauged lagrangian but may include additional invariances of the combined set of Bianchi identities and field equations. Upon partial gauge fixing and integrating out fields one recovers the vector/tensor couplings previously presented in the literature as particular examples of (4.6). We will illustrate this with a few examples in the next section.

The vector/tensor lagrangian (4.6) can be amended by additional matter couplings of the vector fields to scalar and fermion fields

$$\mathcal{L} = \mathcal{L}_{\text{VT}} + \mathcal{L}_{\text{matter}}\,. \tag{4.7}$$

In these matter couplings the electric and magnetic vector fields enter exclusively via the covariant derivatives (3.2) and therefore take a symplectically covariant form. It is important to note that due to (3.10) the covariant derivatives are invariant under tensor gauge transformations.

Under the variations $A_\mu{}^M \to A_\mu{}^M + \delta A_\mu{}^M$, and $B_{\mu\nu\,\alpha} \to B_{\mu\nu\,\alpha} + \delta B_{\mu\nu\,\alpha}$, the vector/tensor lagrangian (4.6) changes as

$$\begin{aligned}\delta\mathcal{L}_{\text{VT}} = & -\frac{1}{8} g \varepsilon^{\mu\nu\rho\sigma}\Big(\Theta^{\Lambda\alpha}\delta B_{\mu\nu\,\alpha} - 2X^\Lambda{}_M{}^\Sigma A_\mu{}^M \delta A_{\nu\,\Sigma} - 2\, X^\Lambda{}_{M\Sigma} A_\mu{}^M \delta A_\nu{}^\Sigma\Big)(\mathcal{H}-\mathcal{G})_{\rho\sigma\,\Lambda} - \\ & -\frac{1}{12} g \varepsilon^{\mu\nu\rho\sigma}\, \delta A_{\mu\,\Lambda}\Big(\Theta^{\Lambda\alpha}\, \mathcal{H}^{(3)}_{\nu\rho\sigma\,\alpha} + 6\, X^\Lambda{}_M{}^\Sigma A_\nu{}^M (\mathcal{H}-\mathcal{G})_{\rho\sigma\,\Sigma}\Big) + \\ & +\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\delta A_\mu{}^\Lambda\Big(\partial_\nu \mathcal{G}_{\rho\sigma\,\Lambda} - g X_{M\Lambda}{}^\Sigma\, A_\nu{}^M \mathcal{G}_{\rho\sigma\Sigma} + g X_{M\Sigma\Lambda}\, A_\nu{}^M \mathcal{H}_{\rho\sigma}{}^\Sigma\Big)\,, \end{aligned} \tag{4.8}$$

with $\mathcal{G}_{\mu\nu\,\Lambda}$ defined in (3.24). From these variations one reads off the equations of motion resulting from (4.7),

$$
\begin{aligned}
g\,\Theta^{\Lambda\alpha}\,(\mathcal{H}-\mathcal{G})_{\mu\nu\,\Lambda} &= 0\,, & (4.9)\\
\frac{1}{12} g\,\varepsilon^{\mu\nu\rho\sigma}\,\Big(\Theta^{\Lambda\alpha}\,\mathcal{H}^{(3)}_{\nu\rho\sigma\,\alpha} + 6\,X^{\Lambda}{}_{M}{}^{\Sigma} A_\nu{}^{M} (\mathcal{H}-\mathcal{G})_{\rho\sigma\,\Sigma}\Big) &= g\,j^{\mu\,\Lambda}\,, & (4.10)\\
-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Big(\partial_\nu \mathcal{G}_{\rho\sigma\,\Lambda} - gX_{M\Lambda}{}^{\Sigma}\,A_\nu{}^{M}\mathcal{G}_{\rho\sigma\Sigma} + gX_{M\Sigma\Lambda}\,A_\nu{}^{M}\mathcal{H}_{\rho\sigma}{}^{\Sigma}\Big) &= g\,j^{\mu}{}_{\Lambda}\,, & (4.11)
\end{aligned}
$$

where $(j^{\mu\Lambda}, j^{\mu}{}_{\Lambda})$ denote the magnetic and electric current densities associated with $\mathcal{L}_{\text{matter}}$, which are defined by

$$
g\,j^{\mu\,\Lambda} = \frac{\delta\mathcal{L}_{\text{matter}}}{\delta A_{\mu\,\Lambda}}\,, \qquad g\,j^{\mu}{}_{\Lambda} = \frac{\delta\mathcal{L}_{\text{matter}}}{\delta A_{\mu}{}^{\Lambda}}\,. \tag{4.12}
$$

Gauge invariance requires these currents to satisfy the following constraints (subject to the matter field equations),

$$
D_\mu\, j^{\mu}{}_{M} = 0\,, \qquad \Theta^{\Lambda\alpha}\, j_{\mu\,\Lambda} = \Theta_{\Lambda}{}^{\alpha}\, j_{\mu}{}^{\Lambda}\,. \tag{4.13}
$$

Equation (4.9) is the duality equation that relates the field strengths $\mathcal{H}_{\rho\sigma\,\Lambda}$ of the magnetic vector fields to the electric field strengths via (3.24), at least for the components projected by $\Theta^{\Lambda\alpha}$. Equation (4.10) relates the relevant tensor field strengths (remember that the components of the tensor field other than $\Theta^{\Lambda\alpha}B_{\mu\nu\,\alpha}$ are not present in the lagrangian and thus do not lead to independent field equations) to the gauge fields and the magnetic matter current.[4] Equations (4.9) and (4.10) thus determine the field strengths of the magnetic vectors and of the tensor fields, respectively, in terms of the other fields. They do not play the role of dynamical field equations, but together with the combined vector and tensor gauge invariances they ensure that the number of propagating degrees of freedom has not changed upon the introduction of tensor and magnetic vector fields in the gauged theory. We will present a more explicit analysis of the degrees of freedom after proper gauge fixing in the next section. At $g = 0$ both (3.24) and (4.10) are identically satisfied which is consistent with the fact that in the ungauged theory the tensor and magnetic vector fields drop from the lagrangian. Finally, (4.11) via (3.24) constitutes the dynamical equations of motion for n vector fields.

5. Applications

In this section we will illustrate a number of features of the general results presented above. The universal lagrangian (4.6) presented in the last section combines tensor fields with electric and magnetic vector fields. We argued above that the total number of degrees of freedom is independent of the embedding tensor, i.e. it remains unchanged with respect to the ungauged theory owing to the fact that magnetic vector and tensor fields appear with their own gauge invariances and couple with a topological first-order kinetic

[4]In the presence of scalar fields, (4.10) takes the form of the duality equation that relates scalar and tensor fields. We shall return to this feature in the next section.

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term. In concrete applications it is often useful to fix most of the gauge invariances and eliminate the auxiliary fields in order to arrive at a formulation in terms of only physical fields. The universal lagrangian (4.6) offers various possibilities of gauge fixing which lead to different effective lagrangians that are related by nonlocal field redefinitions and/or electric/magnetic duality.

Below, in subsection 5.1, we present a general way of gauge fixing by integrating out all the tensor fields from the lagrangian. This leads to an effective lagrangian in terms of n physical vector fields and confirms the analysis of degrees of freedom given above. The result can be interpreted as effecting an electric/magnetic duality transformation directly at the level of the lagrangian. In the next subsection 5.2 we consider a particular class of abelian gaugings generated by translational isometries which are often relevant for the effective field theories that describe flux compactifications. We show that for these gaugings there is an alternative way of gauge fixing which instead leads to a lagrangian in terms of electric vector fields and tensors upon eliminating some of the scalar fields. Finally, in subsection 5.3 we briefly comment on the general results of this paper in the context of $N = 2$ supergravity.

5.1 Gauge fixing

In this subsection we exhibit how the tensor fields can be integrated out from the universal lagrangian (4.6) by choosing a convenient basis for the embedding tensor. Upon further gauge fixing of the remaining tensor gauge transformations this yields a lagrangian containing precisely n physical vector fields. We choose a basis of the magnetic vector fields $A_\mu{}^\Lambda$ and the generators t_α such that the rectangular matrix $\Theta^{\Lambda\alpha}$ decomposes into a square invertible submatrix Θ^{Ii} (with inverse $(\Theta^{-1})_{iI}$), with all other submatrices Θ^{Im}, Θ^{Ui} and Θ^{Um} vanishing. Hence we decomposed the G-generators according to $t_\alpha \to (t_i, t_m)$ and the magnetic vector fields $A_{\mu\Lambda} \to (A_{\mu I}, A_{\mu U})$. Note that the decomposition of the generators and fields is not yet completely fixed, as one can, for instance, redefine the t_i by adding terms linear in the t_m. From (3.4) we deduce the following constraints on the remaining components of the embedding tensor,

$$\Theta_I{}^m = 0\,, \qquad \Theta^{Ii}\,\Theta_I{}^j = \Theta^{Ij}\,\Theta_I{}^i\,. \tag{5.1}$$

In this basis, equation (4.9) takes the form

$$\big(\Theta_I{}^i + \Theta^{Ji}\mathcal{R}_{IJ}\big)\,B_{\mu\nu}{}^I - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\Theta^{Ji}\mathcal{I}_{IJ}\,B^{\rho\sigma\,I} = 2\,\Theta^{Ii}\,\mathcal{J}_{\mu\nu\,I}\,, \tag{5.2}$$

with

$$\begin{aligned}
B_{\mu\nu}{}^I &= g\,\Theta^{Ij}B_{\mu\nu\,j}\,,\\
\mathcal{J}_{\mu\nu\,I} &= \mathcal{F}_{\mu\nu\,I} - \mathcal{R}_{I\Lambda}\mathcal{F}_{\mu\nu}{}^\Lambda + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{I}_{I\Lambda}\mathcal{F}^{\rho\sigma\,\Lambda} + \varepsilon_{\mu\nu\rho\sigma}\mathcal{O}^{\rho\sigma}{}_I\,.
\end{aligned} \tag{5.3}$$

Observe that no other tensor fields will appear in the lagrangian by virtue of (5.1). After some manipulation (5.2) gives rise to

$$(\mathcal{I} + r\,\mathcal{I}^{-1}r)_{IJ}\,B_{\mu\nu}{}^J = \varepsilon_{\mu\nu\rho\sigma}\,\mathcal{J}^{\rho\sigma}{}_I + 2\,(r\,\mathcal{I}^{-1})_I{}^J\,\mathcal{J}_{\mu\nu\,J}\,, \tag{5.4}$$

with $r_{IJ} \equiv \mathcal{R}_{IJ} + (\Theta^{-1})_{iI}\Theta_J{}^i$ a symmetric matrix and $\mathcal{I}_{IK}(\mathcal{I}^{-1})^{KJ} = \delta_I^J$. Substitution of this expression for $B_{\mu\nu}{}^I$ into the lagrangian (4.6) leads to the following terms,

$$\mathcal{L}_{\rm B} = \frac{1}{4}\left[(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}\right]^{IJ}\left[\mathcal{J}_{\mu\nu I}\,\mathcal{J}^{\mu\nu}{}_J - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(r\,\mathcal{I}^{-1})_I{}^K\,\mathcal{J}_{\mu\nu K}\,\mathcal{J}_{\rho\sigma J}\right], \tag{5.5}$$

which should be added to the B-independent terms of the lagrangian (4.6), so that we are dealing with a lagrangian that depends on $2n$ vector fields. However, the magnetic vector fields $A_{\mu\,U}$ are actually absent whereas the tensor gauge transformations can be used to eliminate the electric vector fields $A_\mu{}^I$ from the lagrangian, Eventually one thus arrives at a lagrangian $\mathcal{L}_{\rm V}$ formulated in terms of n vector fields $(A_\mu{}^U, A_{\mu\,I})$ carrying the $2n$ degrees of freedom.

To see that the lagrangian does not depend on the fields $A_{\mu\,U}$, we first observe that neither $\mathcal{J}_{\mu\nu\,I}$ nor the B-independent terms in (4.6) contain the field strengths $\mathcal{F}_{\mu\nu\,U}$. Hence in the abelian case the absence of $A_{\mu\,U}$ is obvious. In the non-abelian case this is less obvious. Although we know that $X^U = 0 = X_{(MN)}{}^U$, this does not exclude that no upper indices U will appear on the generators. Fortunately the absence of $A_{\mu\,U}$ can be directly inferred from the fact that (4.10) is proportional to $\Theta^{\Lambda\alpha}$ which vanishes for $\Lambda = U$ (in particular, the matter current $j^U = 0$), so that we conclude that $\delta\mathcal{L}_{\rm V}/\delta A_{\mu\,U} = 0$. Here it is important to realize that the tensor field equations (4.9) are identically satisfied, so that the variations from $B_{\mu\nu}{}^I$ as defined by (5.4) will not contribute. Note that we are only dealing with the matter currents $j_{\mu\,U}$ and $j_\mu{}^I$ as $j_\mu{}^U = 0$ and $j_{\mu\,I} = (\Theta^{-1})_{iI}\Theta_J{}^i\,j_\mu{}^J$.

We now combine (4.6) and (5.5) to find the new lagrangian $\mathcal{L}_{\rm V}$. For simplicity we evaluate this lagrangian in the abelian case without moment couplings, so that the only quantities involved are the abelian field strengths $\mathcal{F}_{\mu\nu I}$ and $\mathcal{F}_{\mu\nu}{}^U$. The result takes the following form,

$$\begin{aligned}\mathcal{L}_{\rm V} = {}& \frac{1}{4}\Big[\hat{\mathcal{I}}^{IJ}\,\mathcal{F}_{\mu\nu I}\,\mathcal{F}^{\mu\nu}{}_J + \hat{\mathcal{I}}_{UV}\,\mathcal{F}_{\mu\nu}{}^U\,\mathcal{F}^{\mu\nu V} + 2\,\hat{\mathcal{I}}^I{}_U\,\mathcal{F}_{\mu\nu I}\,\mathcal{F}^{\mu\nu U}\Big] + \\ & + \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Big[\hat{\mathcal{R}}^{IJ}\,\mathcal{F}_{\mu\nu I}\,\mathcal{F}_{\rho\sigma J} + \hat{\mathcal{R}}_{UV}\,\mathcal{F}_{\mu\nu}{}^U\,\mathcal{F}_{\rho\sigma}{}^V + 2\,\hat{\mathcal{R}}^I{}_U\,\mathcal{F}_{\mu\nu I}\,\mathcal{F}_{\rho\sigma}{}^U\Big],\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\hat{\mathcal{I}}^{IJ} &= [(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IJ}, \\ \hat{\mathcal{I}}_{UV} &= \mathcal{I}_{UV} + [(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IJ}\Big[\mathcal{R}_{UI}\,\mathcal{R}_{JV} - \mathcal{I}_{UI}\,\mathcal{I}_{JV} - 2\,(r\mathcal{I}^{-1})_J{}^K\,\mathcal{R}_{I(U}\,\mathcal{I}_{V)K}\Big], \\ \hat{\mathcal{I}}^I{}_U &= [(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IJ}\Big[-\mathcal{R}_{JU} + (r\mathcal{I}^{-1})_J{}^K\,\mathcal{I}_{KU}\Big], \\ \hat{\mathcal{R}}^{IJ} &= -\,[(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IK}\,(r\mathcal{I}^{-1})_K{}^J, \\ \hat{\mathcal{R}}_{UV} &= \mathcal{R}_{UV} + [(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IJ}\Big[(-\mathcal{R}_{IU}\,\mathcal{R}_{VK} + \mathcal{I}_{IU}\,\mathcal{I}_{VK})(r\mathcal{I}^{-1})_J{}^K - 2\,\mathcal{R}_{I(U}\,\mathcal{I}_{V)J}\Big], \\ \hat{\mathcal{R}}^I{}_U &= [(\mathcal{I} + r\,\mathcal{I}^{-1}r)^{-1}]^{IJ}\Big[-\mathcal{I}_{JU} + (r\mathcal{I}^{-1})_J{}^K\,\mathcal{R}_{KU}\Big].\end{aligned} \tag{5.7}$$

We note that (in contrast to the situation in odd dimensions) this gauge-fixed lagrangian allows a smooth limit $g \to 0$. At $g = 0$, however, this does not bring back the original lagrangian (2.5) but rather one related to it by electric/magnetic duality. To see this one

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first performs a shift of the generalized theta angle, $\mathcal{R}_{IJ} \to \mathcal{R}_{IJ} + (\Theta^{-1})_{iI}\Theta_J{}^i = r_{IJ}$, followed by a second duality transformation where (2.3) is the unit matrix in the subspace carrying indices U, V, whereas in the subspace carrying the indices I, J it is an off-diagonal transformation with $W = -Z = \mathbf{1}$ (in other words, the typical strong-weak coupling duality). Hence in this formalism one is able to perform duality transformations at the level of the local lagrangian.

5.2 Abelian gaugings

In many situations one is dealing with a group G of symmetries of the ungauged theory that factorizes into two groups, one of which acts exclusively on the matter fields. This situation is, for instance, relevant for abelian gaugings, where the vector fields transform in a trivial representation and the matter fields transform in a non-trivial representation of the (abelian) gauge group. In that case the gauge group can be embedded into a group that acts exclusively on the matter fields. Many supersymmetric models show this feature.

Assuming that the gauge group will be embedded into a rigid invariance group that is decomposable into $\mathrm{G_V} \times \mathrm{G_M}$, where $\mathrm{G_M}$ acts exclusively on the matter fields, we decompose the generators accordingly into two mutually commuting sets: $\{t_\alpha\} = \{t_A\} \oplus \{t_a\}$, where only the generators t_A induce a nontrivial action on the vector fields. The latter implies that the $(t_a)_M{}^N$ vanish as these generators act exclusively in the matter sector. Obviously we are dealing with two sets of structure constants, $f_{AB}{}^C$ and $f_{ab}{}^c$. The embedding tensor $\Theta_M{}^\alpha$ decomposes into $\Theta_M{}^A$ and $\Theta_M{}^a$, which define the gauge group generators $X_M = \Theta_M{}^A t_A + \Theta_M{}^a t_a$. The quadratic constraint (3.3) then decomposes into two separate equations,

$$f_{AB}{}^C\, \Theta_M{}^A\, \Theta_N{}^B + (t_A)_N{}^P\, \Theta_M{}^A \Theta_P{}^C = 0\,, \tag{5.8}$$

$$f_{ab}{}^c\, \Theta_M{}^a\, \Theta_N{}^b + (t_A)_N{}^P\, \Theta_M{}^A \Theta_P{}^c = 0\,. \tag{5.9}$$

The second quadratic constraint (3.7) leads to an additional condition (see also, the comment below (3.7)),

$$\Theta^{\Lambda[a}\Theta_\Lambda{}^{b]} = 0\,. \tag{5.10}$$

For abelian gaugings we have $\Theta_M{}^A = 0$ and the commutativity of the matter charges is ensured by (5.9). The vector/tensor lagrangian for abelian gaugings takes a rather simple form,

$$\begin{aligned}\mathcal{L}_{\rm VT} = {}& \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\,\mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{H}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{H}_{\rho\sigma}{}^\Sigma + \mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{O}^{\mu\nu}{}_\Lambda + \\ & + \frac{1}{2}[\mathcal{I}^{-1}]^{\Lambda\Sigma}\mathcal{O}_{\mu\nu\,\Lambda}\mathcal{O}^{\mu\nu}{}_\Sigma - \frac{1}{4}g\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda a}\, B_{\mu\nu\,a}\,\partial_\rho A_{\sigma\Lambda} + \frac{1}{32}g^2\Theta^{\Lambda\,a}\Theta_\Lambda{}^b\,\varepsilon^{\mu\nu\rho\sigma}\, B_{\mu\nu\,a}\, B_{\rho\sigma\,b}\,,\end{aligned} \tag{5.11}$$

where

$$\mathcal{H}_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu}A_{\nu]}{}^\Lambda + \frac{1}{2}g\,\Theta^{\Lambda\,a}B_{\mu\nu\,a}\,. \tag{5.12}$$

A particular example of abelian gaugings concerns the case of a nonlinear sigma model with gauged translational isometries of its scalar target space. Such gaugings for instance appear in Calabi-Yau (or half-flat manifold) compactifications in the presence of background fluxes [3, 14–18]. Let us thus consider a scalar target space parametrized by scalar

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fields $\{\phi^a, q^i\}$ whose metric G_{mn} does not depend on the subset $\{q^i\}$ of scalar fields such that the shifts $q^i \to q^i + c^i$ constitute a set of abelian isometries. A gauging of these isometries is encoded in an embedding tensor $\Theta_M{}^i = (\Theta_\Lambda{}^i, \Theta^{\Lambda\, i})$ subject to (5.10). It induces the covariant derivatives

$$D_\mu q^i = \partial_\mu q^i - g A_\mu{}^\Lambda \Theta_\Lambda{}^i - g A_{\mu\,\Lambda} \Theta^{\Lambda\, i} \,. \tag{5.13}$$

The magnetic vector fields $\Theta^{\Lambda\, I} A_{\mu\,\Lambda}$ can then be integrated out using the equations of motion (4.10),

$$\varepsilon^{\mu\nu\rho\sigma}\, \partial_\nu B_{\rho\sigma\, i} \;\propto\; G_{ia}(\phi)\, \partial^\mu \phi^a + G_{ij}(\phi) \left(\partial^\mu q^j - g A^{\mu\Lambda} \Theta_\Lambda{}^j - g A^\mu{}_\Lambda \Theta^{\Lambda\, j} \right). \tag{5.14}$$

This shows that the topological term (4.3) eventually gives rise to a topological coupling $\varepsilon^{\mu\nu\rho\sigma}\, \Theta_\Lambda{}^i B_{\rho\sigma\, i}\, \partial_\mu A_\nu{}^\Lambda$ between tensor and electric vector fields as well as to a kinetic term $(G^{-1})^{ij}(\phi)\; \partial_{[\mu} B_{\nu\rho]\, i}\; \partial^{[\mu} B^{\nu\rho]}{}_j$ for the tensor fields. This leads to a lagrangian whose physical fields comprise tensor and electric vector fields, which reproduces the results of [3, 19, 20] Alternatively, following the gauge fixing procedure described in the previous subsection, leads instead to a lagrangian expressed exclusively in terms of (electric and magnetic) vector fields. The general formalism presented here allows rather straightforward generalizations involving the gauging of nonabelian isometries in the presence of tensor fields. Integrating out scalar and magnetic vector fields in the nonabelian case will presumably lead to the non-polynomial interactions of tensor fields captured by (extensions of) the Freedman-Townsend models [21 – 23]. Other applications or generalizations may, for instance, involve M-theory compactifications on twisted tori [24, 25].

5.3 $N = 2$ supersymmetry

As a final topic we briefly discuss gaugings of $N = 2$ supergravity, where the scalar target space is a direct product of a special-Kähler and a quaternion-Kähler manifold whose coordinates we denote by complex fields z^i and real q^u, respectively. The isometry group factors into the direct product $\mathrm{G_{SK}} \times \mathrm{G_Q}$. Only the generators of $\mathrm{G_{SK}}$ induce a nontrivial action on the vector fields. This is a special case of the situation described in the beginning of the previous subsection. Accordingly, we label the generators of the isometry groups as $\{t_\alpha\} = \{t_A\} \oplus \{t_a\}$. A gauging is encoded in an embedding tensor $\Theta_M{}^\alpha = (\Theta_M{}^A, \Theta_M{}^a)$, subject to the constraints (5.9) and (5.10). The kinetic term for the scalar fields is described by a nonlinear sigma-model

$$\mathcal{L}_{\rm kin} = -\frac{1}{2}\, g_{i\bar\jmath}\, D_\mu z^i\, D^\mu \overline{z}^{\bar\jmath} - \frac{1}{2}\, h_{uv}\, D_\mu q^u\, D^\mu q^v \,, \tag{5.15}$$

where $g_{i\bar\jmath}$ and h_{uv} denote the metrics on the special-Kähler and the quaternion-Kähler manifold, respectively, and the covariant derivatives,

$$\begin{aligned} D_\mu z^i &= \partial_\mu z^i - g\, \Theta_M{}^A A_\mu{}^M k^i{}_A \,, \\ D_\mu q^u &= \partial_\mu q^u - g\, \Theta_M{}^a A_\mu{}^M k^u{}_a \,, \end{aligned} \tag{5.16}$$

are written in terms of the embedding tensor and the corresponding Killing vector fields ${k^i}_A$ and ${k^u}_a$. For gaugings that involve only electric vector fields $N = 2$ supersymmetry requires a scalar potential [7 − 9], which can be written as follows,

$$
\begin{aligned}
V = {} & L^\Lambda \bar{L}^\Sigma \Big({\Theta_\Lambda}^A {\Theta_\Sigma}^B \, g_{i\bar{\jmath}} \, {k^i}_A \, {k^{\bar{\jmath}}}_B + 4 \, {\Theta_\Lambda}^a {\Theta_\Sigma}^b \, h_{uv} \, {k^u}_a \, {k^v}_b \Big) + \\
& + \vec{\mathcal{P}}_a \cdot \vec{\mathcal{P}}_b \; {\Theta_\Lambda}^a {\Theta_\Sigma}^b \Big(g^{i\bar{\jmath}} {f_i}^\Lambda \, {\bar{f}_{\bar{\jmath}}}^{\;\Sigma} - 3 \, L^\Lambda \bar{L}^\Sigma \Big) \, .
\end{aligned} \tag{5.17}
$$

Here, L^Λ denotes the upper half of the symplectic section $L^M = (L^\Lambda, M_\Lambda) \equiv e^{\mathcal{K}/2}(X^\Lambda, F_\Lambda)$ on the special Kähler manifold with Kähler potential $\mathcal{K}$, and ${f_i}^\Lambda \equiv (\partial_i + \frac{1}{2}\partial_i \mathcal{K}) \, L^\Lambda$ denotes its Kähler covariant derivative; the Sp(1) vectors $\vec{\mathcal{P}}_a$ are the quaternion-Kähler moment maps associated with the Killing vectors ${k^u}_a$.

It is now straightforward to generalize this expression to a situation where both electric and magnetic vector fields are involved in the gauging. Here we recall that the potential arises as a supersymmetric completion associated with the gauging. However, the electric-magnetic duality plays only an ancillary role in this sector, as is known, for instance, from the gaugings in maximal supergravity theories. There it was demonstrated that the so-called T-tensors are directly expressible in terms of the embedding tensor without the necessity of making a distinction between magnetic and electric components. In fact electric and magnetic components of the embedding tensor can only be identified by referring to the kinetic terms of the vector fields. Hence, the embedding tensor and the T-tensor, and thus the potential (which is quadratic in the T-tensor) is insensitive to these features and does not change under vector-tensor and vector-vector dualities [1, 4, 13].

With the above observations in mind, we may thus write the full scalar potential as a symplectically covariant expression (treating the embedding tensors as a spurionic quantity),

$$
\begin{aligned}
V = {} & L^M \bar{L}^N \Big({\Theta_M}^A {\Theta_N}^B \, g_{i\bar{\jmath}} \, {k^i}_A {k^{\bar{\jmath}}}_B + 4 \, {\Theta_M}^a {\Theta_N}^b \, h_{uv} \, {k^u}_a {k^v}_b \Big) + \\
& + \vec{\mathcal{P}}_a \cdot \vec{\mathcal{P}}_b \; {\Theta_M}^a {\Theta_N}^b \Big(g^{i\bar{\jmath}} {f_i}^M \, {\bar{f}_{\bar{\jmath}}}^{\;N} - 3 \, L^M \bar{L}^N \Big) \, .
\end{aligned} \tag{5.18}
$$

For the abelian case, where ${\Theta_M}^A = 0$, this expression coincides with the one presented long ago in [26]. Of course, a full supersymmetric derivation requires to cast the results of this paper in a supersymmetric context.

Acknowledgments

We wish to thank L. Andrianopoli, F. Saueressig, S. Vandoren and M. Weidner for helpful discussions. H.S. and M.T. thank the CERN theory division for hospitality while this work was started; B.d.W. thanks the Benasque Center for Science, where the work was completed at the String Theory Workshop. This work is partly supported by EU contracts MRTN-CT-2004-005104, MRTN-CT-2004-503369, and MRTN-CT-2004-512194, the INTAS contract 03-51-6346, and the DFG grant SA 1336/1-1.

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